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A generalization of Poisson–Nijenhuis structures

Aïssa Wade

Mathematics Department, University of North Carolina, C.B. #3250, Chapel Hill, NC 27599-3250, USA

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Abstract

We generalize Poisson–Nijenhuis structures. We prove that on a manifold endowed with a Nijenhuis tensor and a Jacobi structure which are compatible, there is a hierarchy of pairwise compatible Jacobi structures. Furthermore, we study the homogeneous Poisson–Nijenhuis structures and their relations with Jacobi structures.

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1. Introduction

Poisson–Nijenhuis manifolds, first introduced by Magri and Morosi [12] in their paper and further studied in [7], play a central role in the study of integrable systems. A Poisson–Nijenhuis structure on a manifold M is given by a pair (π, J) formed by a Poisson tensor π and a $(1, 1)$-tensor field J whose Nijenhuis torsion vanishes, such that for any two differential 1-forms α, β, we have the following compatibility conditions:

$$(\mathcal{C}) \quad \pi(\alpha, {}^{\mathrm{t}}J\beta) = \pi({}^{\mathrm{t}}J\alpha, \beta), \qquad C(\pi, J)(\alpha, \beta) = 0,$$

where ${}^{\mathrm{t}}J$ denotes the transpose of J and $C(\pi, J)$ is an $\mathbb{R}$-bilinear, skew-symmetric operation on the space of differential 1-forms that we shall define below.

In [16], the author defined the Poisson–Nijenhuis structures in the general algebraic framework of Gel'fand and Dorfman. Moreover, Kosmann-Schwarzbach [8] gave a characterization of Poisson–Nijenhuis structures in terms of Lie algebroids. Another characterization is given in [1]. The particular case of a Poisson–Nijenhuis manifold having

E-mail address: aissaw@math.unc.edu (A. Wade).

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a non-degenerate Poisson tensor (or symplectic-Nijenhuis manifold) is especially interesting, this case was studied by several authors for different purposes and under various names (see, e.g., [3]). A natural question arises: what is the odd-dimensional analogue of a symplectic-Nijenhuis manifold? The main aim of the paper is to study this question. A contact manifold is known to be the analogue of a symplectic manifold for the odd-dimensional case. But a natural framework for a unified study of both contact and symplectic manifolds is given by the Jacobi structures. A *Jacobi structure* on a manifold M is defined by a pair (Λ, E), where Λ is a bivector field, E is a vector field such that $[E, \Lambda] = 0$, and $[\Lambda, \Lambda] = 2E \wedge \Lambda$. Jacobi structures were introduced by Lichnerowicz and studied by him and his collaborators [2,4,10] (see also [6]).

In the theory of Hamiltonian systems an important mechanism allows to construct a hierarchy of pairwise compatible Poisson tensors starting from a Poisson–Nijenhuis structure. This brings us to the problem of finding a more general mechanism that would be operational for Jacobi structures. So, we shall need to extend the compatibility conditions $(\mathcal{C})$ above to the case of Jacobi manifolds. Recently, Marrero et al. (see [14]) considered *Jacobi–Nijenhuis structures* and gave a possible solution to the above questions. Here, we present an approach that is slightly different from the one used in [14]. However, these approaches are not equivalent and we shall compare the two approaches in Section 3.2.

This paper is organized as follows. In Section 2, we recall some definitions and basic results concerning Jacobi structures.

In Section 3, we give necessary and sufficient conditions for a $(1, 1)$-tensor field J and a Jacobi structure (Λ, E) to define, in a natural way, a new Jacobi structure which is compatible with (Λ, E) in the sense of Nunes da Costa [15]. This section contains our main results which are Theorems 3.4 and 3.11.

Section 4 is devoted to the analysis of homogeneous Poisson structures, which are compatible with a $(1, 1)$-tensor field J. Such structures are called homogeneous Poisson–Nijenhuis structures. It is well known that homogeneous Poisson structures are related to Jacobi structures, their relations having already been established in [2]. We give sufficient conditions for the existence of a homogeneous Poisson–Nijenhuis structure on a manifold and deduce consequences for the existence and properties of Jacobi structures.

2. Preliminaries

In this paper, all manifolds, multi-vector fields and forms are assumed to be differentiable of class C^∞.

2.1. Jacobi structures

Definition 2.1. A *Jacobi manifold* $(M, \{,\})$ is a manifold M equipped with an $\mathbb{R}$-bilinear, skew-symmetric map $\{,\} : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$, called the *Jacobi bracket*, which satisfies the following properties:

(A) the Jacobi identity:

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0 \quad \forall f,g,h\in C^{\infty}(M,\mathbb{R});$$

(B) the bracket is local (i.e. the support of $\{f,g\}$ is a subset of the intersection of the supports of f and g).

Equivalently, a Jacobi structure can be defined by a pair (Λ, E) of a bivector field Λ and a vector field E such that

$$[E,\Lambda]=0, \qquad [\Lambda,\Lambda]=2E\wedge\Lambda,$$

where [,] is the Schouten–Nijenhuis bracket on the space of multi-vector fields (see [9]). The Jacobi bracket is then given by

$$\{f,g\}=\Lambda(\mathrm{d}f,\mathrm{d}g)+\langle f\,\mathrm{d}g-g\,\mathrm{d}f,E\rangle.$$

When E is zero, we obtain a Poisson structure. In other words, a *Poisson structure* on a manifold M is given by a bivector field Λ such that the Schouten–Nijenhuis bracket $[\Lambda,\Lambda]$ vanishes. Then (M,Λ) is called a *Poisson manifold*. In [10], Lichnerowicz showed that to any Jacobi structure (Λ, E) on a manifold M, one may associate a Poisson structure π on $M\times\mathbb{R}$, defined by

$$\pi(x,t)=\mathrm{e}^{-t}\left(\Lambda(x)+\frac{\partial}{\partial t}\wedge E\right).$$

Then, π is called the *Poissonization* of (Λ, E).

Notations. To any bivector field Λ on M, we may associate the skew-symmetric linear map denoted also by $\Lambda:\Omega^1(M)\to\chi(M)$ and defined by

$$\langle\beta,\Lambda\alpha\rangle=\langle\alpha\wedge\beta,\Lambda\rangle=\Lambda(\alpha,\beta),$$

where $\chi(M)$ is the space of vector fields and $\Omega^1(M)$ is the space of differential 1-forms on M. Conversely, a linear map $\Lambda:\Omega^1(M)\to\chi(M)$ defines a bivector field on M if and only if

$$\langle\alpha,\Lambda\beta\rangle+\langle\beta,\Lambda\alpha\rangle=0.$$

Example (Locally conformal symplectic manifolds). Let M be a $2n$-dimensional manifold. A *locally conformal symplectic structure* on M is given by a pair (F,ω), where F is a non-degenerate 2-form and ω is a 1-form such that

$$\mathrm{d}\omega=0, \qquad \mathrm{d}F+\omega\wedge F=0.$$

We define a bivector field Λ and a vector field E by

$$i_E F=\omega, \qquad i_{\Lambda\alpha}F=-\alpha \quad \forall\alpha\in\Omega^1(M).$$

Then (Λ, E) defines a Jacobi structure. For any $x\in M$, there exist a neighborhood U_x and a function f defined on U_x such that $\omega=\mathrm{d}f$ and $\Omega=\mathrm{e}^f F$ is symplectic.

2.2. The Lie algebroid of a Jacobi manifold

It was proven in [5] that there is a Lie algebroid associated with an arbitrary Jacobi manifold (M, Λ, E). We refer the reader to [11], for instance, for the basic properties of Lie algebroids. Consider the vector bundle $T^*M \oplus \mathbb{R}$. The space $\Gamma(T^*M \oplus \mathbb{R})$ of smooth sections may be identified with $\Omega^1(M) \times C^\infty(M, \mathbb{R})$. The Lie algebroid associated with a Jacobi manifold (M, Λ, E) is $T^*M \oplus \mathbb{R}$ with the Lie bracket $\{,\}_{(\Lambda,E)}$ on $\Gamma(T^*M \oplus \mathbb{R})$ defined by

$$\begin{aligned}\{(\alpha, f), (\beta, g)\}_{(\Lambda,E)} = {} & (L_{\Lambda\alpha}\beta - L_{\Lambda\beta}\alpha - \mathrm{d}(\Lambda(\alpha, \beta)) + fL_E\beta - gL_E\alpha - i_E(\alpha \wedge \beta), \\ & -\Lambda(\alpha, \beta) + \Lambda(\alpha, \mathrm{d}g) - \Lambda(\beta, \mathrm{d}f) + fE(\mathrm{d}g) - gE(\mathrm{d}f)),\end{aligned}$$

where d is the exterior derivative and $L_X = \mathrm{d}i_X + i_X\mathrm{d}$ is the Lie derivation by X, for any vector field X. The anchor is given by the map $\#_{(\Lambda,E)}$ such that

$$\#_{(\Lambda,E)}(\alpha, f) = \Lambda\alpha + fE.$$

3. Jacobi and Nijenhuis structures

3.1. Extension of the definition of compatibility to Jacobi structures

Let J be a (1, 1)-tensor field of M. The *Nijenhuis torsion* N_J of J with respect to the Lie bracket $[\cdot, \cdot]$ on the space $\chi(M)$ of vector fields is defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] \quad \forall X, Y \in \chi(M).$$

Definition 3.1. J is called a *Nijenhuis tensor* if its Nijenhuis torsion vanishes.

In particular, when J is a (1, 1)-tensor field on M and $\Lambda : \Omega^1(M) \to \chi(M)$ is a linear map, then $J \circ \Lambda$ defines a bivector field if and only if $J \circ \Lambda = \Lambda \circ {}^{\mathrm{t}}J$. In this case, the associated bivector field is denoted by $J\Lambda$.

Furthermore, any bivector field Λ gives rise to a bracket defined on the differential 1-forms by

$$\{\alpha, \beta\}_\Lambda = L_{\Lambda\alpha}\beta - L_{\Lambda\beta}\alpha - \mathrm{d}(\Lambda(\alpha, \beta)) \quad \forall \alpha, \beta \in \Omega^1(M), \tag{1}$$

where L_X is the Lie derivation by X for any vector field X.

Whenever $J \circ \Lambda = \Lambda \circ {}^{\mathrm{t}}J$, we denote by $C(\Lambda, J)$ the $\mathbb{R}$-bilinear map given by

$$C(\Lambda, J)(\alpha, \beta) = \{\alpha, \beta\}_{J\Lambda} - (\{{}^{\mathrm{t}}J\alpha, \beta\}_\Lambda + \{\alpha, {}^{\mathrm{t}}J\beta\}_\Lambda - {}^{\mathrm{t}}J\{\alpha, \beta\}_\Lambda).$$

Definition 3.2 (see [7]). A Poisson–Nijenhuis structure on a manifold M is defined by a Poisson tensor π and a Nijenhuis tensor J on M such that:

1. $J \circ \pi = \pi \circ {}^{\mathrm{t}}J$,

2. $C(\pi, J) = 0$.

In this case, we say that π and J are compatible.

An extension of this definition to Jacobi structures is given by the following definition.

Definition 3.3. Let (M, Λ, E) be a Jacobi manifold and let J be a $(1, 1)$-tensor field such that its Nijenhuis torsion N_J satisfies

$$N_J(\Lambda\alpha, \Lambda\beta) = 0, \qquad N_J(\Lambda\alpha, J^k E) = 0$$

for all $\alpha, \beta \in \Omega^1(M)$ and for all $k \in \mathbb{N}$. Then, J is said to be compatible with the Jacobi structure (Λ, E) if:

(i) $J \circ \Lambda = \Lambda \circ {}^{\mathrm{t}}J$,
(ii) $\langle {}^{\mathrm{t}}J\gamma, \Lambda(C(\Lambda, J)(\alpha, \beta)) - \Lambda(\alpha, \beta)JE + \Lambda(\alpha, {}^{\mathrm{t}}J\beta)E\rangle = 0$ for any $\alpha, \beta, \gamma \in \Omega^1(M)$,
(iii) $[J^k E, \Lambda] + [E, J^k\Lambda] = 0$ for any integer $k \geq 1$.

When $E = 0$ (i.e. Λ defines a Poisson structure), the definition reduces to that of a *weak Poisson–Nijenhuis structure* (see [13]). The compatibility conditions then reduce to (i) and (ii) and constitute a generalization of the definition of a Poisson–Nijenhuis structure introduced in [12].

Theorem 3.4. *Let (Λ, E) be a Jacobi structure on M. Assume that J is a $(1, 1)$-tensor field such that $J \circ \Lambda = \Lambda \circ {}^{\mathrm{t}}J$,*

$$N_J(\Lambda\alpha, \Lambda\beta) = 0, \qquad N_J(\Lambda\alpha, E) = 0 \quad \forall\alpha, \beta \in \Omega^1(M),$$

where N_J is the Nijenhuis torsion of J. Then $(J\Lambda, JE)$ is a Jacobi structure on M if and only if the following properties are satisfied for all $\alpha, \beta, \gamma \in \Omega^1(M)$:

(a) $J([JE, \Lambda]\alpha + [E, J\Lambda]\alpha) = 0$,
(b) $\langle {}^{\mathrm{t}}J\gamma, \Lambda(C(\Lambda, J)(\alpha, \beta)) - \Lambda(\alpha, \beta)JE + \Lambda(\alpha, {}^{\mathrm{t}}J\beta)E\rangle = 0$.

In particular, if J is a Nijenhuis tensor compatible with (Λ, E), then $(J\Lambda, JE)$ is a Jacobi structure on M.

The proof of the above theorem is based on the following three lemmas.

Lemma 3.5. *For any bivector field Λ,*

$$\langle\gamma, \Lambda\{\alpha, \beta\}_\Lambda\rangle = \langle\gamma, [\Lambda\alpha, \Lambda\beta]\rangle + \tfrac{1}{2}[\Lambda, \Lambda](\alpha, \beta, \gamma) \quad \forall\alpha, \beta, \gamma \in \Omega^1(M). \tag{2}$$

This formula is proven for instance in [7].

Lemma 3.6. *Consider a pair (Λ, E) of a bivector field Λ and a vector field E on M such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$. Then, for any $C^\infty(M, \mathbb{R})$-linear map J on $\chi(M)$ satisfying*

$J \circ \Lambda = \Lambda \circ {}^{t}J$, *the following formula holds:*

$$\tfrac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) = (JE \wedge J\Lambda)(\alpha,\beta,\gamma) + \langle {}^{t}J\gamma, \Lambda(C(\Lambda, J)(\alpha,\beta))\rangle \\ + E({}^{t}J\gamma)\Lambda(\alpha, {}^{t}J\beta) - JE({}^{t}J\gamma)\Lambda(\alpha,\beta) - \langle\gamma, N_J(\Lambda\alpha, \Lambda\beta)\rangle.$$

Proof. Lemma 3.5 yields

$$\tfrac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) = \langle {}^{t}J\gamma, \Lambda\{\alpha,\beta\}_{J\Lambda}\rangle - \langle\gamma, [J\Lambda\alpha, J\Lambda\beta]\rangle.$$

A direct computation, using relation (2) again, shows that

$$\tfrac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) = \tfrac{1}{2}([\Lambda,\Lambda]({}^{t}J\alpha, \beta, {}^{t}J\gamma) + [\Lambda,\Lambda](\alpha, {}^{t}J\beta, {}^{t}J\gamma) \\ - [\Lambda,\Lambda](\alpha,\beta, {}^{t}J^{2}\gamma)) + \langle {}^{t}J\gamma, \Lambda(C(\Lambda, J)(\alpha,\beta))\rangle \\ - \langle\gamma, N_J(\Lambda\alpha, \Lambda\beta)\rangle. \quad (3)$$

Since $[\Lambda,\Lambda] = 2E \wedge \Lambda$, we obtain

$$\tfrac{1}{2}[J\Lambda, J\Lambda](\alpha,\beta,\gamma) = (JE \wedge J\Lambda)(\alpha,\beta,\gamma) + E({}^{t}J\gamma)\Lambda(\alpha, {}^{t}J\beta) - JE({}^{t}J\gamma)\Lambda(\alpha,\beta) \\ + \langle {}^{t}J\gamma, \Lambda(C(\Lambda, J)(\alpha,\beta))\rangle - \langle\gamma, N_J(\Lambda\alpha, \Lambda\beta)\rangle.$$

This is the formula that we sought. □

Lemma 3.7. *Let Λ be a bivector field and E a vector field on M. The following relation holds for any $C^{\infty}(M, \mathbb{R})$-linear map J on $\chi(M)$:*

$$[JE, J\Lambda](\alpha,\beta) = \langle\beta, N_J(E, \Lambda\alpha)\rangle + \langle\beta, J[JE, \Lambda]\alpha + J[E, J\Lambda]\alpha - J^{2}[E, \Lambda]\alpha\rangle.$$

Proof. For any bivector field Λ and for all $\alpha, \beta \in \Omega^{1}(M)$, we have

$$[E, \Lambda](\alpha,\beta) = L_E(\Lambda(\alpha,\beta)) - \Lambda(L_E\alpha, \beta) - \Lambda(\alpha, L_E\beta).$$

This is equivalent to the relation

$$[E, \Lambda]\alpha = [E, \Lambda\alpha] - \Lambda L_E\alpha \quad \forall\alpha \in \Omega^{1}(M). \quad (4)$$

Using (4), we obtain for any $\alpha \in \Omega^{1}(M)$:

$$[JE, J\Lambda]\alpha = [JE, J\Lambda\alpha] - J\Lambda L_{JE}\alpha \\ = N_J(E, \Lambda\alpha) + J[JE, \Lambda\alpha] + J[E, J\Lambda\alpha] - J^{2}[E, \Lambda\alpha] - J\Lambda L_{JE}\alpha.$$

Replacing $[E, \Lambda\alpha]$ by $[E, \Lambda]\alpha + \Lambda L_E\alpha$, we deduce that

$$[JE, J\Lambda]\alpha = N_J(E, \Lambda\alpha) + J([JE, \Lambda\alpha] - \Lambda L_{JE}\alpha) + J([E, J\Lambda\alpha] - J\Lambda L_E\alpha) \\ - J^{2}[E, \Lambda]\alpha = N_J(E, \Lambda\alpha) + J([JE, \Lambda] + [E, J\Lambda] - J[E, \Lambda])\alpha. \quad \square$$

Proof of Theorem 3.4. Lemma 3.7 ensures that $[JE, J\Lambda] = 0$ is equivalent to (a), while Lemma 3.6 asserts that $[J\Lambda, J\Lambda] = 2JE \wedge J\Lambda$ if and only if Property (b) is satisfied. Therefore, the theorem is proved. □

Now, let us express Properties (a) and (b) of Theorem 3.4 using the Lie algebroid bracket associated with the Jacobi structure.

Proposition 3.8. *Let* (Λ, E) *be a Jacobi structure on M and let J be a* $(1, 1)$*-tensor field such that*

$$J \circ \Lambda = \Lambda \circ {}^{t}J, \qquad N_J(\Lambda\alpha + fE, \Lambda\beta + gE) = 0 \quad \forall \alpha, \beta \in \Omega^1(M),$$
$$\forall f, g \in C^\infty(M, \mathbb{R}).$$

Then,

(a) *is satisfied* $\Leftrightarrow [J\Lambda\alpha + fJE, gJE] = \#_{(J\Lambda, JE)}(\{(\alpha, f), (0, g)\}_{(J\Lambda, JE)})$;

(b) *is satisfied* $\Leftrightarrow [J\Lambda\alpha, J\Lambda\beta] = \#_{(J\Lambda, JE)}(\{(\alpha, 0), (\beta, 0)\}_{(J\Lambda, JE)})$.

Proof. On one hand,

$$[J\Lambda\alpha + fJE, gJE] = g[J\Lambda\alpha, JE] + (J\Lambda(\alpha, \mathrm{d}g) + \langle f\,\mathrm{d}g - g\,\mathrm{d}f, JE\rangle)JE.$$

On the other hand,

$$\#_{(J\Lambda, JE)}\{(\alpha, f), (0, g)\}_{(J\Lambda, JE)} = -gJ\Lambda L_{JE}\alpha + (J\Lambda(\alpha, \mathrm{d}g) + \langle f\,\mathrm{d}g - g\,\mathrm{d}f, JE\rangle)JE.$$

We deduce that

$$[J\Lambda\alpha + fJE, gJE] - \#_{(J\Lambda, JE)}\{(\alpha, f), (0, g)\}_{(J\Lambda, JE)}$$
$$= g([J\Lambda\alpha, JE] + J\Lambda L_{JE}\alpha) = g[J\Lambda, JE]\alpha.$$

But Lemma 3.7 asserts that

$$[J\Lambda, JE]\alpha = 0 \Leftrightarrow J([JE, \Lambda]\alpha + [E, J\Lambda]\alpha) = 0.$$

Hence we obtain the first equivalence. In the same way, we prove the second equivalence using Lemma 3.6. □

3.2. Hierarchy of Jacobi structures

A manifold M is said to be a *bi-Hamiltonian manifold* if M is endowed with two Poisson tensors π_1 and π_2 such that $\pi_1 - \lambda\pi_2$ is a Poisson tensor for any $\lambda \in \mathbb{R}$. Then $\pi_1 - \lambda\pi_2$ is called a *Poisson pencil*. By analogy, if $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are two Jacobi structures such that $\{\cdot, \cdot\}_\lambda = \{\cdot, \cdot\}_1 - \lambda\{\cdot, \cdot\}_2$ defines a Jacobi structure for any λ in $\mathbb{R}$, then $\{\cdot, \cdot\}_\lambda$ will be called a *Jacobi pencil*. In this case, the two Jacobi structures are said to be *compatible* (see [15]).

Proposition 3.9. *Let* (Λ_1, E_1) *and* (Λ_2, E_2) *be two Jacobi structures on M. Denote by* $\pi_i = \mathrm{e}^{-t}(\Lambda_i + (\partial/\partial t) \wedge E_i)$ *with* $i = 1, 2$, *the associated Poisson tensors on* $M \times \mathbb{R}$. *Then the following assertions are equivalent:*

1. (Λ_1, E_1) *and* (Λ_2, E_2) *define a Jacobi pencil on M.*

2. $[E_1, \Lambda_2] + [E_2, \Lambda_1] = 0$ *and* $[\Lambda_1, \Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1$.
3. *The pair* (π_1, π_2) *defines a Poisson pencil on* $M \times \mathbb{R}$.

The proof of this proposition is straightforward and can be found in [15].

Remark 3.10. Consider the following relation:

(a$'$) $[JE, \Lambda] + [E, J\Lambda] = 0$.

Under the hypotheses of Theorem 3.4, if $(J\Lambda, JE)$ is a Jacobi structure compatible with (Λ, E), then it follows from the proposition above and Theorem 3.4 that (a$'$), as well as (b) are satisfied.

On the other hand, if (a$'$) and (b) are satisfied then Theorem 3.4 ensures that $(J\Lambda, JE)$ is a Jacobi structure on M. But $(J\Lambda, JE)$ may not be compatible with (Λ, E). They define a Jacobi pencil if and only if $[J\Lambda, \Lambda] = JE \wedge \Lambda + E \wedge J\Lambda$.

Theorem 3.11. *For any Jacobi structure* (Λ, E) *compatible with a Nijenhuis tensor J on M and for each* $k \in \mathbb{N}^*$, *the pair* $(J^k\Lambda, J^kE)$ *is a Jacobi structure on M. Furthermore, for* $k_1, k_2 \in \mathbb{N}^*$, $(J^{k_1}\Lambda, J^{k_1}E)$ *and* $(J^{k_2}\Lambda, J^{k_2}E)$ *define a Jacobi pencil.*

This theorem is a generalization of a result proved in [7,12].

Lemma 3.12. *Let J be a* $(1, 1)$-*tensor field. Then, we have*

$$\begin{aligned} N_{J^{k+1}}(X, Y) = {} & N_{J^k}(JX, JY) + J^k(N_J(J^kX, Y) + N_J(X, J^kY)) \\ & -J^2(N_{J^{k-1}}(JX, JY) - N_{J^k}(X, Y)) \quad \forall X, Y \in \chi(M). \end{aligned}$$

The proof of this lemma is straightforward.

Proof of Theorem 3.11. Assume that $[J^\ell\Lambda, J^\ell\Lambda] = 2J^\ell E \wedge J^\ell\Lambda$ for any $\ell \leq k$. It follows from Lemma 3.6 that

$$\begin{aligned} & \tfrac{1}{2}[J^{k+1}\Lambda, J^{k+1}\Lambda](\alpha, \beta, \gamma) \\ & \quad = (J^{k+1}E \wedge J^{k+1}\Lambda)(\alpha, \beta, \gamma) + \langle {}^{\mathrm{t}}J\gamma, J^k\Lambda C(J^k\Lambda, J)(\alpha, \beta)\rangle \\ & \qquad + J^kE({}^{\mathrm{t}}J\gamma)J^k\Lambda(\alpha, {}^{\mathrm{t}}J\beta) - J^{k+1}E({}^{\mathrm{t}}J\gamma)J^k\Lambda(\alpha, \beta) \end{aligned}$$

for all α, β, γ in $\Omega^1(M)$. We shall prove that

$$J^k\Lambda C(J^k\Lambda, J)(\alpha, \beta) + J^k\Lambda(\alpha, {}^{\mathrm{t}}J\beta)J^kE - J^k\Lambda(\alpha, \beta)J^{k+1}E = 0$$

for any $k \geq 1$. In fact, for any bivector field Λ and for any linear map J on $\chi(M)$ such that $J \circ \Lambda = \Lambda \circ {}^{\mathrm{t}}J$, the following relation holds (see [12]):

$$\langle C(J\Lambda, J)(\alpha, \beta), X\rangle = \langle C(\Lambda, J)({}^{\mathrm{t}}J\alpha, \beta), X\rangle + \langle \alpha, N_J(\Lambda\beta, X)\rangle. \tag{5}$$

For any vector field X of the form $\Lambda\gamma$, we have

$$\langle C(J\Lambda, J)(\alpha, \beta), \Lambda\gamma\rangle = \langle C(\Lambda, J)({}^{\mathrm{t}}J\alpha, \beta), \Lambda\gamma\rangle.$$

Thus,

$$\Lambda C(J\Lambda, J)(\alpha, \beta) = \Lambda C(\Lambda, J)({}^{t}J\alpha, \beta)$$

for all α, β in $\Omega^1(M)$. Hence, we obtain by induction that for any $k \geq 1$,

$$J^k \Lambda C(J^k\Lambda, J)(\alpha, \beta) = J^k \Lambda C(\Lambda, J)({}^{t}J^k\alpha, \beta). \tag{6}$$

Since J is compatible with (Λ, E), we have

$$J\Lambda C(\Lambda, J)(\alpha, \beta) = J(\Lambda(\alpha, \beta)JE - \Lambda(\alpha, {}^{t}J\beta)E). \tag{7}$$

We deduce that

$$\begin{aligned} J^k \Lambda C(J^k\Lambda, J)(\alpha, \beta) &= J^k \Lambda C(\Lambda, J)({}^{t}J^k\alpha, \beta) \\ &= J^k(\Lambda({}^{t}J^k\alpha, \beta)JE - \Lambda({}^{t}J^k\alpha, {}^{t}J\beta)E) \\ &= J^k\Lambda(\alpha, \beta)J^{k+1}E - J^k\Lambda(\alpha, {}^{t}J\beta)J^kE \end{aligned}$$

for any $k \geq 1$. So, we obtain the relation that we sought. The latter implies that

$$[J^k\Lambda, J^k\Lambda] = 2J^kE \wedge J^k\Lambda \quad \text{for any } k \geq 1.$$

Moreover, replacing J by J^k in Lemma 3.7, we obtain since $[E, \Lambda] = 0$,

$$[J^kE, J^k\Lambda](\alpha, \beta) = \langle \beta, N_{J^k}(E, \Lambda\alpha)\rangle + \langle {}^{t}J^k\beta, [J^kE, \Lambda]\alpha + [E, J^k\Lambda]\alpha\rangle.$$

From Lemma 3.12, we obtain by induction that $N_{J^k}(E, \Lambda\alpha)$ vanishes for all $k \geq 1$. Therefore,

$$[J^kE, J^k\Lambda] = 0 \quad \text{for all } k \geq 1.$$

Thus $(J^k\Lambda, J^kE)$ defines a Jacobi structure for any $k \geq 1$.

Now, take two different pairs $(J^{k_1}\Lambda, J^{k_1}E)$ and $(J^{k_2}\Lambda, J^{k_2}E)$. We shall prove that they determine a Jacobi pencil. For any $\lambda \in \mathbb{R}$, we have to prove that

$$[J^{k_1}\Lambda - \lambda J^{k_2}\Lambda, J^{k_1}\Lambda - \lambda J^{k_2}\Lambda] = 2(J^{k_1}E - \lambda J^{k_2}E) \wedge (J^{k_1}\Lambda - \lambda J^{k_2}\Lambda).$$

We already know that

$$[J^{k_i}\Lambda, J^{k_i}\Lambda] = 2J^{k_i}E \wedge J^{k_i}\Lambda \quad \forall i = 1, 2.$$

Now we prove that

$$[J^{k_1}\Lambda, J^{k_2}\Lambda] = J^{k_1}E \wedge J^{k_2}\Lambda + J^{k_2}E \wedge J^{k_1}\Lambda.$$

Assume that $k_1 = k_2 + \ell$, then we apply ℓ times the result saying that for arbitrary bivector fields Λ and π on M, for any linear map J on $\chi(M)$ the following formula holds (see [12]):

$$\begin{aligned} [J\Lambda, \pi](\alpha, \beta, \gamma) &= [\Lambda, \pi](\alpha, \beta, {}^{t}J\gamma) + \langle C(\pi, J)(\alpha, \gamma), \Lambda\beta\rangle \\ &\quad - \langle C(\pi, J)(\beta, \gamma), \Lambda\alpha\rangle - \langle C(\Lambda, J)(\alpha, \beta), \pi\gamma\rangle. \end{aligned} \tag{8}$$

We apply this last relation and we calculate by recursion the ℓ quantities $[J^{k_2+\ell}\Lambda, J^{k_2}\Lambda], \dots, [J^{k_2+1}\Lambda, J^{k_2}\Lambda]$. It follows that:

$$\begin{aligned}
&[J^{k_1}\Lambda, J^{k_2}\Lambda](\alpha,\beta,\gamma)\\
&\quad= [J^{k_2}\Lambda, J^{k_2}\Lambda](\alpha,\beta,{}^{\mathrm t}J^{\ell}\gamma) + \sum_{r=1}^{\ell}\langle C(J^{k_2}\Lambda, J)(\alpha, {}^{\mathrm t}J^{r-1}\gamma), J^{k_1-r}\Lambda\beta\rangle\\
&\qquad -\sum_{r=1}^{\ell}\langle C(J^{k_2}\Lambda, J)(\beta, {}^{\mathrm t}J^{r-1}\gamma), J^{k_1-r}\Lambda\alpha\rangle\\
&\qquad -\sum_{r=1}^{l}\langle C(J^{k_1-r}\Lambda, J)(\alpha,\beta), J^{k_2+r-1}\Lambda\gamma\rangle.
\end{aligned}$$

Using (6) and (7) and the fact that $[J^{k_2}\Lambda, J^{k_2}\Lambda] = 2J^{k_2}E \wedge J^{k_2}\Lambda$, we obtain after computations

$$[J^{k_1}\Lambda, J^{k_2}\Lambda] = J^{k_1}E \wedge J^{k_2}\Lambda + J^{k_2}E \wedge J^{k_1}\Lambda.$$

The last step is to show that

$$[J^{k_1}E - \lambda J^{k_2}E, J^{k_1}\Lambda - \lambda J^{k_2}\Lambda] = 0.$$

This is equivalent to showing that $[J^{k_1}E, J^{k_2}\Lambda] + [J^{k_2}E, J^{k_1}\Lambda] = 0$. By hypothesis this relation is true when $k_2 = 1$ and using Lemma 3.7, we can easily show by induction that this formula holds for any k_1 and k_2. □

Example. Let ω be a closed 1-form and let F_1, F_2 be two non-degenerate 2-forms on M. Assume that (F_1,ω) and (F_2,ω) are locally conformal symplectic structures on M. Let (Λ_i, E_i) denote the Jacobi structures associated with (F_i,ω), where $i = 1, 2$. Assume that these two Jacobi structures are compatible. Define the isomorphism of $C^{\infty}(M,\mathbb{R})$-modules $\flat_i : \chi(M) \to \Omega^1(M)$ by

$$\flat_i(X) = -i_X F_i.$$

We have

$$E_i = -\flat_i^{-1}(\omega), \qquad \Lambda_i\alpha = \flat_i^{-1}(\alpha) \quad \forall\alpha \in \Omega^1(M).$$

Then, the (1, 1)-tensor field $J = \flat_2^{-1} \circ \flat_1$ is compatible with (Λ_1, E_1). Indeed, for any $x \in M$, there exist a neighborhood U_x and a function f defined on U_x such that $\omega = \mathrm{d}f$. The 2-forms $\Omega_1 = \mathrm{e}^f F_1$ and $\Omega_2 = \mathrm{e}^f F_2$ are symplectic and the Poisson tensors associated with Ω_1 and Ω_2 are, respectively, $\pi_1 = \mathrm{e}^{-f}\Lambda_1$ and $\pi_2 = \mathrm{e}^{-f}\Lambda_2$.

We claim that the Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are compatible if and only if π_1 and π_2 are compatible. Let us prove this claim. Using the properties of the Schouten–Nijenhuis bracket, we get

$$[\pi_1, \pi_2] = \mathrm{e}^{-2f}([\Lambda_1, \Lambda_2] - [\Lambda_1, f] \wedge \Lambda_2 - [\Lambda_2, f] \wedge \Lambda_1).$$

Since $E_i = [\Lambda_i, f] = -\Lambda_i(\mathrm{d}f)$, we have

$$[\pi_1, \pi_2] = \mathrm{e}^{-2f}([\Lambda_1, \Lambda_2] - E_1 \wedge \Lambda_2 - E_2 \wedge \Lambda_1).$$

Therefore $[\pi_1, \pi_2] = 0$ if and only if $[\Lambda_1, \Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1$. Moreover, we may remark that the Jacobi identity for the Schouten–Nijenhuis bracket yields

$$\begin{aligned}[][[\pi_1, \pi_2], \mathrm{e}^f] &= -[[\pi_2, \mathrm{e}^f], \pi_1] - [[\mathrm{e}^f, \pi_1], \pi_2] \\ &= -[[\Lambda_2, f], \mathrm{e}^{-f}\Lambda_1] - [[f, \Lambda_1], \mathrm{e}^{-f}\Lambda_2].\end{aligned}$$

The fact that $E_i = [\Lambda_i, f]$ implies

$$[[\pi_1, \pi_2], \mathrm{e}^f] = -\mathrm{e}^{-f}([E_2, \Lambda_1] + [E_1, \Lambda_2]).$$

Thus $(\Lambda_i, E_i)_{i=1,2}$ form a Jacobi pencil if and only if the tensors $(\pi_i)_{i=1,2}$ define a Poisson pencil. So, we may deduce that the Nijenhuis torsion of J vanishes. Furthermore, the sequence $(J^k\pi_1)$ consists of pairwise compatible Poisson tensors, while $(J^k\Lambda_1, J^kE_1)$ is a sequence of pairwise compatible Jacobi structures.

3.3. Compatibility conditions and Jacobi–Nijenhuis structures

In this section, we shall study the differences between a Jacobi–Nijenhuis structure and a structure which satisfies the axioms of Definition 3.3. We now recall the definition of a Jacobi–Nijenhuis structure given in [14]. For any bivector field Λ and for any vector field E, we may define a map $\tilde{\#}_{(\Lambda,E)} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \chi(M) \times C^\infty(M, \mathbb{R})$ by

$$\tilde{\#}_{(\Lambda,E)}(\alpha, g) = (\Lambda\alpha + gE, -i_E\alpha) \quad \forall\alpha \in \Omega^1(M)\ \ \forall g \in C^\infty(M, \mathbb{R}).$$

Consider the Lie bracket $[\cdot,\cdot]_\star$ defined on $\chi(M) \times C^\infty(M, \mathbb{R})$ by

$$[(X_1, f_1), (X_2, f_2)]_\star = ([X_1, X_2], i_{X_1}\mathrm{d}f_2 - i_{X_2}\mathrm{d}f_1).$$

Let $\mathcal{J} : \chi(M) \times C^\infty(M, \mathbb{R}) \to \chi(M) \times C^\infty(M, \mathbb{R})$ be a $C^\infty(M, \mathbb{R})$-linear map. The Nijenhuis torsion of $\mathcal{J}$ is defined as follows:

$$\begin{aligned}N_{\mathcal{J}}((X_1, f_1), (X_2, f_2)) &= [\mathcal{J}(X_1, f_1), \mathcal{J}(X_2, f_2)]_\star - \mathcal{J}[\mathcal{J}(X_1, f_1), (X_2, f_2)]_\star \\ &\quad -\mathcal{J}[(X_1, f_1), \mathcal{J}(X_2, f_2)]_\star + \mathcal{J}^2[(X_1, f_1), (X_2, f_2)]_\star.\end{aligned}$$

Then $\mathcal{J}$ is said to be a *recursion operator* of a Jacobi structure (Λ, E) (see [14]) if

$$N_{\mathcal{J}}(\tilde{\#}_{(\Lambda,E)}(\alpha_1, g_1), \tilde{\#}_{(\Lambda,E)}(\alpha_2, g_2)) = 0$$

for any $(\alpha_i, g_i) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$, $i = 1, 2$.

Definition 3.13. Let $\mathcal{J} : \chi(M) \times C^\infty(M, \mathbb{R}) \to \chi(M) \times C^\infty(M, \mathbb{R})$ be a $C^\infty(M, \mathbb{R})$-linear map, and let (Λ, E) be a Jacobi structure on M. The triple $(\Lambda, E, \mathcal{J})$ is said to be a Jacobi–Nijenhuis structure on M if $\mathcal{J}$ is a recursion operator of (Λ, E), if

$\mathcal{J}\circ\tilde{\#}_{(\Lambda,E)}=\tilde{\#}_{(\Lambda,E)}\circ{}^{\mathrm{t}}\mathcal{J}$ and if (Λ_1,E_1) is a Jacobi structure compatible with (Λ,E), where Λ_1 and E_1 are characterized by

$$\tilde{\#}_{(\Lambda_1,E_1)}=\mathcal{J}\circ\tilde{\#}_{(\Lambda,E)}.$$

In fact, a $C^\infty(M,\mathbb{R})$-linear map $\mathcal{J}$ from $\chi(M)\times C^\infty(M,\mathbb{R})$ into itself is equivalent to determining a quadruplet $(J,X_0,\alpha_0,\varphi_0)$, where J is a (1, 1)-tensor field on M, X_0 is a vector field, α_0 is a differential 1-form and φ_0 is a smooth function such that

$$\mathcal{J}(X,f)=(JX+fX_0,i_X\alpha_0+f\varphi_0).$$

We set $\mathcal{J}=(J,X_0,\alpha_0,\varphi_0)$, the transpose ${}^{\mathrm{t}}\mathcal{J}$ of $\mathcal{J}$ is defined by

$${}^{\mathrm{t}}\mathcal{J}(\beta,g)=({}^{\mathrm{t}}J\beta+g\alpha_0,i_{X_0}\beta+g\varphi_0)$$

for any $(\beta,g)\in\Omega^1(M)\times C^\infty(M,\mathbb{R})$. For any Jacobi structure (Λ,E),

$$\mathcal{J}\circ\tilde{\#}_{(\Lambda,E)}=\tilde{\#}_{(\Lambda,E)}\circ{}^{\mathrm{t}}\mathcal{J}\Leftrightarrow\begin{cases}i_E\alpha_0=0,\\ JE=\Lambda\alpha_0+\varphi_0E,\\ J\Lambda\alpha-\Lambda{}^{\mathrm{t}}J\alpha=(i_E\alpha)X_0+(i_{X_0}\alpha)E\quad\forall\alpha\in\Omega^1(M).\end{cases}$$

From now on, we shall assume that $i_E\alpha_0=0$.

We can now compare Jacobi–Nijenhuis structures with structures defined by a (1, 1)-tensor field J and a Jacobi structure which are compatible in the sense of Definition 3.3.

We start with the simple example where $\Lambda=0$ (i.e. the Jacobi structure is just a non-zero vector field on M), then for any (1, 1)-tensor field J, the properties (i)–(iii) of Definition 3.3 are satisfied. Taking into account the above equivalence, we deduce that a recursion operator $\mathcal{J}=(J,X_0,\alpha_0,\varphi_0)$ of $(0,E)$ defines a Jacobi–Nijenhuis structure on M if and only if $JE=\varphi_0E$ and X_0 vanishes on the support of E. In other words, any (1, 1)-tensor field J is compatible with $(0,E)$ but for a recursion operator $\mathcal{J}=(J,X_0,\alpha_0,\varphi_0)$, the pair $(E,\mathcal{J})$ is a Jacobi–Nijenhuis structure on M if and only if the following conditions are fulfilled:

$$JE=\varphi_0E,\qquad X_0=0\quad\text{on}\ \mathrm{supp}(E).$$

We now move on to the general case. Let $\mathcal{J}=(J,X_0,\alpha_0,\varphi_0)$ be a recursion operator of a Jacobi structure (Λ,E), where Λ is not identically zero on M. There are two alternatives:

First case. $X_0=0$ on the support of E. We shall show that in such a case $(\Lambda,E,\mathcal{J})$ is a Jacobi–Nijenhuis structure if and only if $JE=\Lambda\alpha_0+\varphi_0E$ and J fulfills the axioms (i)–(iii) of Definition 3.3.

Assume that $(\Lambda,E,\mathcal{J})$ is a Jacobi–Nijenhuis structure. By hypothesis, we have (i) and the equality $JE=\Lambda\alpha_0+\varphi_0E$. Moreover, for every integer $k\geq 1$, the equation

$$\tilde{\#}_{(\Lambda_k,E_k)}=\mathcal{J}^k\circ\tilde{\#}_{(\Lambda,E)}$$

has a unique solution that is $(\Lambda_k,E_k)=(J^k\Lambda,J^kE)$. A result proven in [14] shows that if $(\Lambda,E,\mathcal{J})$ is a Jacobi–Nijenhuis structure then for any integer $k\geq 1$, the pair $(J^k\Lambda,J^kE)$

is a Jacobi–Nijenhuis structure that is compatible with (Λ, E). So, we obtain in particular (iii). Furthermore, setting

$$\mathcal{V} = \{x \in M | E(x) = 0 \text{ in a neighborhood of } x\},$$

it follows from the fact that $\mathcal{J}$ is a recursion operator that the Nijenhuis torsion N_J of J satisfies the following equation on $\text{supp}(E) \cup \mathcal{V}$:

$$N_J(\Lambda\alpha + fE, \Lambda\beta + gE) = 0. \tag{9}$$

Since $\text{supp}(E) \cup \mathcal{V}$ is a dense subset of M, by an argument of continuity, Eq. (9) is valid on the whole manifold M. Furthermore, $JE = \Lambda\alpha_0 + \varphi_0 E$ implies that $J^k E$ is also of the form $\Lambda\alpha + \varphi E$, for all $k \geq 1$. So,

$$N_J(\Lambda\alpha, J^k E) = 0 \quad \forall \alpha \in \Omega^1(M).$$

Applying Theorem 3.4, we obtain (ii). This shows that J belongs to the particular class of structures satisfying Definition 3.3 and $JE = \Lambda\alpha_0 + \varphi_0 E$.

Conversely, assume that the relations $JE = \Lambda\alpha_0 + \varphi_0 E$, (i)–(iii) are satisfied on M. Then using Theorem 3.11 and Definition 3.13, we deduce that $(\Lambda, E, \mathcal{J})$ is a Jacobi–Nijenhuis structure.

Second case. If X_0 is not identically zero on the support of E, then $J \circ \Lambda - \Lambda \circ {}^{\mathrm{t}}J \neq 0$. However $(\Lambda, E, \mathcal{J})$ may be a Jacobi–Nijenhuis structure.

We summarize our discussion in the following proposition.

Proposition 3.14. *Let (M, Λ, E) be a Jacobi manifold and let J be a $(1, 1)$-tensor field on M such that $J \circ \Lambda = \Lambda \circ {}^{\mathrm{t}}J$. Assume that there exist a vector field X_0, a 1-form α_0 and a smooth function φ_0 such that $\mathcal{J} = (J, X_0, \alpha_0, \varphi_0)$ is a recursion operator of (Λ, E). Then $(\Lambda, E, \mathcal{J})$ is a Jacobi–Nijenhuis structure if and only if the following conditions are satisfied:*

(C_1) $JE = \Lambda\alpha_0 + \varphi_0 E$,
(C_2) $[J^k E, \Lambda] + [E, J^k \Lambda] = 0$ *for any integer* $k \geq 1$,
(C_3) $\langle {}^{\mathrm{t}}J\gamma, \Lambda(C(\Lambda, J)(\alpha, \beta)) - \Lambda(\alpha, \beta)JE + \Lambda(\alpha, {}^{\mathrm{t}}J\beta)E\rangle = 0$ *for any* $\alpha, \beta, \gamma \in \Omega^1(M)$.

4. Nijenhuis tensors and homogeneous Poisson structures

Definition 4.1, A homogeneous Poisson manifold (M, π, Z) is a Poisson manifold (M, π) with a vector field Z over M such that

$$[Z, \pi] = -\pi.$$

Theorem 4.2. *Assume that (M, π, Z) is a homogeneous Poisson manifold. Let J be Nijenhuis tensor compatible with π. Then $(M, J\pi, Z)$ is a homogeneous Poisson manifold if and only if the following property is satisfied:*

$$\pi \circ (L_Z \circ {}^{\mathrm{t}}J - {}^{\mathrm{t}}J \circ L_Z) = 0, \tag{10}$$

where $L_Z = i_Z \mathrm{d} + \mathrm{d} i_Z$ *is the Lie derivation by Z. When this property holds,* $J\pi - \lambda\pi$ *defines a Poisson pencil which is homogeneous with respect to Z.*

Proof. Taking into account Theorem 3.11, we have only to prove that $[Z, J\pi] = -J\pi$. Let us compute $[Z, J\pi]$. We obtain

$$\begin{aligned}[Z, J\pi](\mathrm{d}f, \mathrm{d}g) &= L_Z(J\pi(\mathrm{d}f, \mathrm{d}g)) - J\pi(L_Z\,\mathrm{d}f, \mathrm{d}g) - J\pi(\mathrm{d}f, L_Z\,\mathrm{d}g) \\ &= L_Z(\pi({}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g)) - \pi({}^{\mathrm{t}}JL_Z\,\mathrm{d}f, \mathrm{d}g) - \pi({}^{\mathrm{t}}J\,\mathrm{d}f, L_Z\,\mathrm{d}g).\end{aligned}$$

Since

$$L_Z(\pi({}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g)) = [Z, \pi]({}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g) + \pi(L_Z{}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g) + \pi({}^{\mathrm{t}}J\,\mathrm{d}f, L_Z\mathrm{d}g),$$

we obtain

$$\begin{aligned}[Z, J\pi](\mathrm{d}f, \mathrm{d}g) &= [Z, \pi]({}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g) + \pi(L_Z{}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g) - J\pi(L_Z\,\mathrm{d}f, \mathrm{d}g) \\ &= -\pi({}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g) + \pi(L_Z{}^{\mathrm{t}}J\,\mathrm{d}f, \mathrm{d}g) - J\pi(L_Z\,\mathrm{d}f, \mathrm{d}g).\end{aligned}$$

Hence, the relation $[Z, J\pi] = -J\pi$ is equivalent to the following one:

$$\pi \circ L_Z \circ {}^{\mathrm{t}}J = \pi \circ {}^{\mathrm{t}}J \circ L_Z.$$

This proves the theorem. □

Definition 4.3. A homogeneous Poisson manifold (M, π, Z) equipped with a Nijenhuis tensor J which is compatible with π and satisfies equation (10) is said to be a homogeneous Poisson–Nijenhuis manifold.

Corollary 4.4. *Let* (M, π, J) *be a Poisson–Nijenhuis manifold. If* π *is homogeneous with respect to a vector field Z and if the following property holds*:

$$[Z, JX] = J[Z, X] \quad \forall X \in \chi(M), \tag{11}$$

then the triple (M, π, J) *is a homogeneous Poisson–Nijenhuis manifold with respect to Z.*

Proof. We obtain this corollary using the above theorem and the fact that

$$[Z, JX] = J[Z, X] \quad \forall X \in \chi(M) \Leftrightarrow L_Z \circ {}^{\mathrm{t}}J = {}^{\mathrm{t}}J \circ L_Z.$$ □

Definition 4.5. A map $\psi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ between two Jacobi manifolds is said to be a *conformal Jacobi morphism* if there exists a function $a \in C^\infty(M_1, \mathbb{R})$ which vanishes nowhere such that for any $f, g \in C^\infty(M_2, \mathbb{R})$, we have

$$\{a(f \circ \psi), a(g \circ \psi)\}_1 = a(\{f, g\}_2 \circ \psi),$$

where the brackets $\{\}_1$ and $\{\}_2$ are the Jacobi brackets associated with (Λ_1, E_1) and (Λ_2, E_2), respectively.

Homogeneous Poisson manifolds are closely related to Jacobi manifolds and their relations were established in [2]. In terms of Poisson pencils, we have the following results.

Proposition 4.6. *Let $\{\cdot,\cdot\}_\lambda$ be a Jacobi pencil on M. There exists a Poisson pencil on $M\times\mathbb{R}$ such that the projection $P: M\times\mathbb{R}\to M$ is a conformal Jacobi morphism for each λ.*

Proof. If (Λ_i, E_i) denotes the Jacobi structure on M associated to $\{\cdot,\cdot\}_i$ with $i=1,2$, then according to Proposition 3.9, the Poisson pencil on $M\times\mathbb{R}$ is given by $\pi_1-\lambda\pi_2$, where

$$\pi_i(x,t)=\mathrm{e}^{-t}\left(\Lambda_i(x)+\frac{\partial}{\partial t}\wedge E_i\right).$$

One may easily verify that $P:(M\times\mathbb{R},\pi_\lambda)\to(M,\{\cdot,\cdot\}_\lambda)$ is a conformal Jacobi morphism. □

Conversely, we may prove that homogeneous Poisson pencils give Jacobi pencils by using a proof from Dazord [2]. More precisely, we have the following proposition.

Proposition 4.7. *Let π_λ be a homogeneous Poisson pencil on M with respect to the vector field Z, and let N be a sub-manifold of M of codimension* 1 *which is transverse to Z. Then there exists a Jacobi pencil on N such that for any pair of functions (f,g) defined on an open set U of M, satisfying $\langle Z,\mathrm{d}f\rangle=f$ and $\langle Z,\mathrm{d}g\rangle=g$, we have*

$$\{f_{|N\cap U}, g_{|N\cap U}\}_\lambda=\pi_\lambda(\mathrm{d}f,\mathrm{d}g)_{|N\cap U}.$$

Corollary 4.8. *Let (M,Λ,E) be a Jacobi manifold and let J be a Nijenhuis tensor on M, which is compatible with (Λ,E). Then there exists a sequence of Poisson–Nijenhuis structures (π_k) on $M\times\mathbb{R}$ such that the projection $P_k:(M\times\mathbb{R},\pi_k)\to(M,\Lambda,E)$ is a conformal Jacobi morphism for each $k\geq 1$.*

Conversely, if (M,π,J) is a homogeneous Poisson–Nijenhuis manifold with respect to the vector field Z and if N is a sub-manifold of M of codimension 1, *which is transverse to Z, then there exists a sequence of pairwise compatible Jacobi structures on N determined by π, Z and J.*

This corollary is a direct consequence of Theorem 3.11 and Propositions 4.6 and 4.7.

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